लक्ष्यं विश्वामानम्

WEST BENGAL STATE UNIVERSITY

B.Sc. Honours/Programme 4th Semester Examination, 2022

MTMHGEC04T/MTMGCOR04T-MATHEMATICS (GE4/DSC4)

Time Allotted: 2 Hours | Full Marks: 50

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.
All symbols are of usual significance.

Answer Question No. 1 and any *five* from the rest

1. Answer any ***five*** questions from the following: $2\times 5 = 10$

(a) In $\mathbb{Z}_{14}$, find the smallest positive integer n such that $n[6]=[0]$. 2

(b) Let $(G,*)$ be a group. If every element of G has its own inverse then prove that G is commutative. 2

(c) Let H be a subgroup of a group G. Show that for all $a\in G$, $aH=H$ if and only if $a\in H$. 2

(d) Check whether the relation ρ defined by $x\rho y$ if and only if $|x|=|y|$, is an equivalence relation or not on the set of integers $\mathbb{Z}$. Justify your answer. 2

(e) Show that the alternative group A_3 is a normal subgroup of S_3. 2

(f) Show that every cyclic group is abelien. 2

(g) Show that the ring of matrices $\left\{\begin{pmatrix} 2a & 0 \\ 0 & 2b \end{pmatrix} : a, b\in\mathbb{Z}\right\}$ contains divisors of zero and does not contain the unity. 2

(h) Let A and B be two ideals of a ring R. Is $A\cup B$ an ideal of R? Justify. 2

2. (a) A relation ρ on the set $\mathbb{N}$ is given by $\rho=\{(a,b)\in \mathbb{N}\times\mathbb{N} : a \text{ is a divisor of } b\}$. Examine if ρ is (i) reflexive, (ii) symmetric, (iii) transitive. 4

(b) If G is a group such that $(ab)^2=a^2b^2$ for all $a,b\in G$; then show that G is commutative. 4

3. (a) Let $A=\{1,2,3\}$. List all one-one functions from A onto A. 4

(b) Let G be a commutative group. Show that the set H of all elements of finite order is a subgroup of G. 4

4. (a) Let H be a subgroup of a group G. Show that the relation ρ defined on G by "$a\rho b$ if and only if $a^{-1}b\in H$" for $a,b\in G$ is an equivalence relation. 4

(b) Prove that the order of every subgroup of a finite group G is a divisor of the order of G. 4

5. (a) Prove that every group of order less than 6 is commutative. 4

(b) Let $(G, \circ)$ be a cyclic group generated by a. Then prove that a^{-1} is also a generator. 4

6. (a) Show that the intersection of two normal subgroups of a group G is normal in G. 4

(b) Show that if H be a subgroup of a commutative group G then the quotient group G/H is commutative. Is the converse true? Justify. 4

7. (a) Prove that an infinite cyclic group has only two generators. 4

(b) In the rings $\mathbb{Z}_8$ and $\mathbb{Z}_6$, find the following elements: 2+2

(i) the units and (ii) the zero divisors.

8. (a) Find all ideals of $\mathbb{Z}$. 4

(b) Let R be a commutative ring with 1. Then prove that R is a field if and only if R has no non-zero proper ideals. 4

9. (a) (i) Let S be a set with n elements. How many binary operations can be defined on S? Justify. 2+2

(ii) Let A and B be two sets with $|A| = 5$ and $|B| = 2$. How many surjective functions defined from A onto B? Justify.

(b) Let $G = \left\{ \begin{pmatrix} a & a \\ a & a \end{pmatrix} : a\,(\neq 0) \in \mathbb{R} \right\}$. Show that G forms a group w.r.t. matrix multiplication. 4

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WEST BENGAL STATE UNIVERSITY

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MTMHGEC04T/MTMGCOR04T-MATHEMATICS (GE4/DSC4)

Time Allotted: 2 Hours | Full Marks: 50

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All symbols are of usual significance.

Answer Question No. 1 and any *five* from the rest

1. Answer any ***five*** questions from the following: 2×5 = 10

(a) Show that the set of cube roots of unity forms a group with respect to multiplication.

(b) In a group $(G, \circ)$ prove that for all $a, b \in G$, $(a \circ b)^{-1} = b^{-1} \circ a^{-1}$.

(c) When a relation ρ defined on a nonempty set S is said to be an equivalence relation?

(d) Prove that in a commutative group G, the set $H = \{x \in G : x = x^{-1}\}$ forms a subgroup of G.

(e) Show that the group $(Z_4, +)$ is a cyclic group. Find it's generators.

(f) Let R be a ring with 1. Show that the subset $T = \{n1 : n \in \mathbb{Z}\}$ is a subring of R.

(g) Show that the ring $(Z_5, +, -)$ is an integral domain.

(h) Determine whether the permutation f on the group S_5 is odd or even where

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 3 & 1 & 2 \end{pmatrix}$$

(i) Define index of a subgroup H of a group G. If $G = S_3$ and $H = A_3$, then find the value of $[G:H]$.

2. (a) Let a relation R defined on the set $\mathbb{Z}$ by "aRb if and only if $a-b$ is divisible by 5" for all $a, b \in \mathbb{Z}$. Show that R is an equivalence relation. 4

(b) Which of the following mathematical systems is / are group(s)? 2+2

(i) $(\mathbb{N}, *)$, where $a * b = a$ for all $a, b \in \mathbb{N}$.

(ii) $(\mathbb{Z}, *)$, where $a * b = a - b$ for all $a, b \in \mathbb{Z}$.

3. (a) Let the permutations f and g are the elements of S_5 where 2+2+1

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 3 & 5 & 1 \end{pmatrix}, \quad g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 4 & 1 \end{pmatrix}.$$ Find fg, gf, f^{-1}.

(b) Let $f: Z \to Z$ is defined by $f(n) = n^2$, $n \in Z$ and $g: Z \to Z$ is defined by $g(n) = 2n$, $n \in Z$. Find the composition of the functions $f \circ g$ and $g \circ f$. 2+1

4. (a) Verify the statement is true or false: In ring R if $(a+b)^2 = a^2 + 2ab + b^2$ for all $a, b \in R$, then R is a commutative ring. 3

(b) (i) Show that the set $S = \left\{ \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} \right\}$, $x \neq 0$ is a subgroup of the group of all 2×2 order non-singular real matrices. 2

(ii) Let $(G, \circ)$ be a commutative group and $H = \{a^2 : a \in G\}$, prove that H is sub-group of G. 3

5. (a) Prove that every subgroup of a cyclic group is cyclic. 4

(b) Let G be a group of prime order. Then prove that G is cyclic. 4

6. (a) Find all right cosets of the subgroup $6\mathbb{Z}$ in the group $(\mathbb{Z}, +)$. 4

(b) Let G be a group such that every cyclic subgroup of G is a normal subgroup of G. Prove that every subgroup of G is a normal subgroup of G. 4

7. (a) Let H be the set of all real matrices 4

$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1 \right\}$. Prove that H is a subset of $GL(2, \boldsymbol{R})$.

(b) Find all cyclic subgroups of the group $(S, \cdot)$, where $S = \{1, i, -1, -i\}$. 4

8. (a) Examine if the ring of matrices $\left\{ \begin{pmatrix} a & b \\ 2b & a \end{pmatrix} : a, b \in \boldsymbol{R} \right\}$ is a field. 4

(b) Prove that a finite integral domain is a field. 4

9. (a) Show that the ring of matrices $\left\{ \begin{pmatrix} 2a & 0 \\ 0 & 2b \end{pmatrix} : a, b \in \mathbf{Z} \right\}$ contains divisors of zeros and does not contain the unity. 4

(b) Prove that the ring $(Z_n, +, \cdot)$ is an integral domain if and only if n is prime. 4

10.(a) Show that $T = \{[0], [5]\}$ is a subring of the ring $\mathbb{Z}_{10}$. 4

(b) Let I and J be ideals of a ring R. Prove that $I + J$ is an ideal of R. 4

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WEST BENGAL STATE UNIVERSITY

B.Sc. Honours/Programme 4th Semester Supplementary Examination, 2021

MTMHGEC04T/MTMGCOR04T-MATHEMATICS (GE4/DSC4)

Time Allotted: 2 Hours | Full Marks: 50

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Candidates are required to give their answers in their own words as far as practicable.
All symbols are of usual significance.

Answer Question No. 1 and any *five* from the rest

1. Answer any ***five*** questions from the following: 2×5 = 10

(a) Write down the power set of the set $\{x,\ y,\ z\}$.

(b) Two mappings $f: R \to R$ and $g: R \to R$ are defined by $f(x) = x^2$; $g(x) = 2x + 3$ for all $x \in R$. Find the mapping $f \circ g$.

(c) Let $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}$ and $g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$ be two permutations. Then find gf.

(d) Prove that $(Z,\ \cdot)$ is not a group, where Z is the set of integers.

(e) State Lagrange's theorem.

(f) Find the generators of the cyclic group $\{\mathbb{Z}, +\}$.

(g) Prove that under the matrix addition and multiplication, the set $M = \left\{ \begin{pmatrix} \alpha & \alpha \\ 0 & \alpha \end{pmatrix} :\ \alpha \in Z \right\}$ is not a ring.

(h) Prove that if every element of a group $(G, \circ)$ be its own inverse then it is an abelian group.

2. (a) Show that a group $(G, *)$ is commutative if and only if $(a * b)^2 = a^2 * b^2$ for all $a \in G$ [where $x^2 = x * x$]. 5

(b) Let $(G, \circ)$ be a group. Prove that $(a * b)^{-1} = a^{-1} * b^{-1}$ for all $a, b \in G$. 3

3. (a) Let $(G, \circ)$ be a group. A non-empty subset H of G forms a subgroup of $(G, \circ)$ if and only if $a \in H,\ b \in H \Rightarrow a \circ b^{-1} \in H$. 4

(b) Let $S = \{1,\ \omega,\ \omega^2\}$, where ω is an imaginary cube root of 1. Prove that S is an abelian group with respect to multiplication. 4

4. (a) Show that the set of even integers forms a commutative ring with respect to the usual addition and multiplication of integers. 5

   (b) If a ring $(R, +, \cdot)$, $a^2 = a$ for all $a \in R$; prove that $a + a = 0$ for all $a \in R$; (0 is the zero element of R). 3

5. (a) In a ring $(R, +, \cdot)$ show that $(-a)\cdot(-b) = a\cdot b$ for all $a, b \in R$. 3

   (b) Let $M = \left\{ \begin{pmatrix} 2a & 0 \\ 0 & 2b \end{pmatrix} : a, b \in Z \right\}$, then show that M is a ring with respect to matrix addition and multiplication. 5

6. (a) Prove that if G is commutative, then every subgroup of G is normal. 4

   (b) Let G be a group and H be a subgroup of G. If $h \in H$ then prove that $hH = H$. 4

7. (a) Prove that every proper subgroup of a group of order 6 is cyclic. 4

   (b) The intersection of two normal subgroups of a group G is a normal subgroup of G. 4

8. (a) Show that $H = \left\{ \begin{pmatrix} a+ib & c+id \\ -c+id & a-ib \end{pmatrix} : a, b, c, d \in Z \right\}$ forms a ring with unity. 4

   (b) Prove that a finite integral domain is a field. 4

9. (a) If f is real function defined by $f(x) = \dfrac{x-1}{x+1}$, then prove that $f(2x) = \dfrac{3f(x)+1}{f(x)+3}$. 3

   (b) If $f(x) = x^2$, then find the value of $\dfrac{f(1.1) - f(1)}{1.1 - 1}$. 2

   (c) Prove that a group $(G, \circ)$ contains only one identity element. 3

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